TRINITY COLLEGE

YEAR 12
MATHEMATICS
SPECIALIST

Test 4, 2023
Section Two: Calculator Allowed
Motion and Statistics

STUDENT'S NAME: Solutions [LAWRENCE]

DATE: Thursday 7th September **TIME:** 50 minutes **MARKS:** 48
ASSESSMENT %: 10

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser
Special Items: 1 A4 page notes, Classpad, Scientific Calculator

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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Question 1 **(8 marks)**

A particle is traveling in a straight line with an initial velocity of 3m/s and an acceleration of $a(t) = 9e^{3t}$.

$t = 0 \quad v = 3$

(a) Determine the distance travelled in the first 3 seconds. (4 marks)

$v(t) = \int 9e^{3t}$

$= 3e^{3t} + c$

$3 = 3e^{0} + c$

$c = 0$

$v(t) = 3e^{3t}$

$\therefore$ Distance $= \int_0^3 |3e^{3t}| \, dt$

$= 8102.08$ metres

✓ $v(t)$
✓ c for $v(t)$
✓ correct integral for distance
✓ distance

(b) Determine an expression for acceleration in terms of displacement, $a(x)$. (2 marks)

$x(t) = \int 3e^{3t}$

$= e^{3t} + C$

$x(t) - c = e^{3t}$

$a = 9e^{3t}$

$a = 9(x - c)$

✓ $x(t)$
✓ $a(x)$

c) Hence, or otherwise, determine the starting position $x(0)$ of the object if, at the 3 second $t=3$ point, the magnitude of the displacement is half the magnitude of the acceleration. (2 marks)

$x(3) = \dfrac{a(3)}{2}$

$x(3) = \dfrac{9e^{3(3)}}{2}$

$x(3) = 36463.87767$

$x(t) = e^{3t} + c$

$36463.87767 = e^{3(3)} + c$

$C = 28360.79375$

$x(t) = e^{3t} + 28360.79$

$x(0) = 1 + 28360.79$

$= 28361.79$

✓ $x(3)$
✓ $x(0)$

Question 2 **(8 marks)**

Researchers found the reaction time of mice by playing a loud buzzing sound which stopped when the mouse turned around. The reaction times were said to be normally distributed with an estimated mean of 151 milliseconds and a standard deviation of σ milliseconds. Samples of 40 mouse's reaction times are recorded and the means are found to create the sample mean distribution $\bar{X}$.

(a) State the distribution for $\bar{X}$ and its parameters. (3 marks)

$n \geq 30$ $\bar{x}$ is normal

$$\therefore \bar{X} \sim N\left(151, \left(\frac{\sigma}{\sqrt{40}}\right)^2\right)$$

✓ stating $\bar{x}$ normal with reason

✓ $\bar{x}$ mean

✓ $\bar{x}$ s.d. (or variance)

(b) Discuss the effects that increasing the sample size would have on $\bar{X}$. (1 mark)

It would decrease $\bar{X}$'s standard deviation / variance

OR

$\bar{X}$ distribution would get skinnier ✓ correct effect

After many samples had been gathered, it was noted that 75% of the samples had a total duration of buzzing of 6000 milliseconds or greater. 0.75

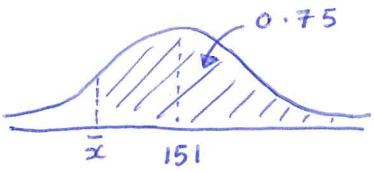
✓ $\bar{x} = 150$ms

✓ s_x

✓ σ

(c) Calculate σ correct to 2 decimal places. (3 marks)

$\bar{x} = \frac{6000}{40} = 150$ms



$\therefore$ solve(normCDF[150, ∞, x, 151]) = 0.75 (σ)

$x = 1.4826022$

$\frac{\sigma}{\sqrt{40}} = 1.4826022$ $\therefore \sigma = 9.38$ms

A researcher is tasked with calculating a confidence interval to capture the true population mean. The researcher stated: "I am going to use a very small sample size for my confidence interval as that will increase its width and hence increase my chances of capturing the population mean".

(d) Explain an issue with the researcher's comment. (1 mark)

- Lower sample size means lower precision / variation so the width needs to be larger to compensate. Does NOT increase confidence.

✓ valid issue

OR

- Due to inherent nature of sampling, not one CI has a higher chance over another to contain the true mean.

Question 3 **(11 marks)**

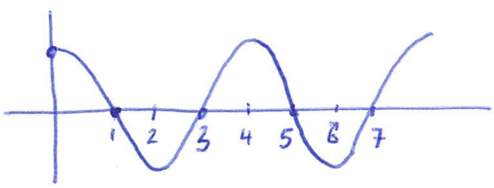
A particle is travelling in a straight line under simple harmonic motion. It is observed only when it passes through the point $x = 0$ periodically at $t = 1, 3, 5, 7, 9, \dots$ seconds.

(a) Discuss what the viewer would observe about the particle's velocity each time it is seen at the point $x = 0$. (2 marks)

Same speed but alternating directions

✓ same speed
✓ alternating directions

(b) Explain why the function $x(t) = A cos\left(\frac{\pi}{2}t\right), A \in \mathbb{R}$ should be used to model this particle's motion. A brief sketch can be used. (2 marks)



At $x = 0$ after a half cycle,
at max/min at $t = 0$

Period $= 4$ $\quad 4 = \frac{2\pi}{\omega}$

$\omega = \frac{2\pi}{4} = \frac{\pi}{2}$

$\therefore x(t) = A\cos\left(\frac{\pi}{2}t\right)$

✓ reason for cos (diagram or wording)
✓ show $w = \pi/2$ (proof)

It is first observed with a velocity of $10m/s$.

$t = 1 \quad v = 10$

(c) Determine an expression for the particle's displacement in terms of time, $x(t)$. (2 marks)

$x = A\cos\left(\frac{\pi}{2}t\right)$

$v = -A\frac{\pi}{2}\sin\left(\frac{\pi}{2}t\right)$

$10 = -A\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)$

$= -A\frac{\pi}{2}(1)$

$10 = -A\frac{\pi}{2}$

$A = -\frac{20}{\pi}$

$x(t) = -\frac{20}{\pi}\cos\left(\frac{\pi}{2}t\right)$

✓ $v(t)$ with A
✓ solve for A & $x(t)$

(d) Prove that the object undergoes simple harmonic motion. (2 marks)

$x(t) = \frac{-20}{\pi} \cos\left(\frac{\pi}{2} t\right)$

$v(t) = 10 \sin\left(\frac{\pi}{2} t\right)$

$a(t) = 10\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2} t\right)$

$= 5\pi \cos\left(\frac{\pi}{2} t\right)$

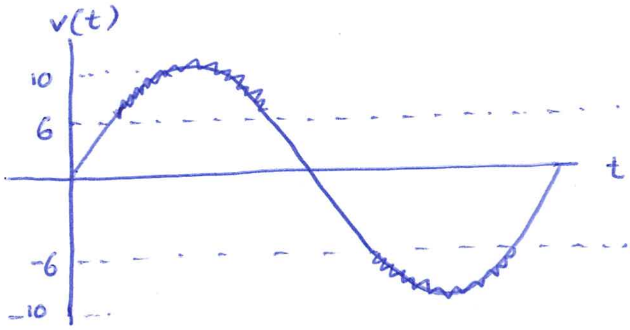
$= \frac{-\pi^2}{4}\left(\frac{-20}{\pi} \cos\left(\frac{\pi}{2} t\right)\right)$

$= \frac{-\pi^2}{4} x(t)$ $\therefore$ SHM

✓ $a(t)$ (from $x(t), v(t)$)

✓ valid proof showing $A(t) = -w^2 x(t)$

(e) Determine the percentage of time this particle spent travelling faster than $6m/s$. (3 marks)



$v > 6$

* Look at 1 full cycle.

solve $v(t) = \pm 6$

$t = 0.41, 1.59, 2.41, 3.59$

$\frac{(1.59 - 0.41) + (3.59 - 2.41)}{4} \times 100$

$\frac{2.36}{4} \times 100$

$= 59\%$ of the time.

✓ solve for t when $v(t) = \pm 6$

✓ working including 4 pts for 1 period or 2 x 2pts for half a period.

✓ correct %

Question 4 **(13 marks)**

In a court of law, a tomato sauce company, NATRAL, is being sued for the inclusion of too much salt within their product. NATRAL claims that their tomato sauce is naturally created and so the salt content may vary per bottle, but they promise an average of 4.50g per bottle.

Lawyer 1, the lawyer suing NATRAL, sampled 100 bottles randomly, with a mean of 4.888g and a sample standard deviation of 0.1728g.

(a) Comment, by creating and using a 99.9% confidence interval, on the promise made by NATRAL. (4 marks)

$99\% \Rightarrow z = 3.2905$

✓ z for 99% CI

$$4.888 - 3.2905\left(\frac{0.1728}{\sqrt{100}}\right) \le \bar{x} \le 4.888 + 3.2905\left(\frac{0.1728}{\sqrt{100}}\right)$$

✓ working for CI
✓ correct CI
✓ correct commen (including not certain)

99% CI : $[4.83,\ 4.94]$

Whilst we cannot say whether or not the μ is in fact higher than 4.50g the sample does appear to be taken from a population with a mean higher than 4.50g.

(b) Calculate the sample size required to reduce the 99.9% confidence interval's width to below 0.06. (2 marks)

± 0.03

$$4.888 + 0.03 = 4.888 + 3.2905\left(\frac{0.1728}{\sqrt{n}}\right)$$

$$0.03 = 3.2905\left(\frac{0.1728}{\sqrt{n}}\right)$$

✓ correct formula including n

$n = 359.2$

✓ rounding up.

$\therefore n = 360$ scores in sample.

Lawyer 2, the lawyer in charge of defending NATRAL, comes up with a cunning plan. This lawyer took the original sample of 100 bottles and split it into two smaller samples of 50.

Table 1

	Sample's Mean	Sample Standard Deviation	Lower limit of confidence interval	Higher Limit of confidence interval	Confidence level
Sample A, size 50	a	0.0756g	4.7180g	4.7600g	b
Sample B, size 50	5.034g	c	5.0074g	5.0606g	95%

(c) Determine the values of a, b and c. (1, 2, 2 = 5 marks)

$a = \frac{4.7600 + 4.7180}{2}$

$\therefore a = 4.739$

✓ for a

$4.76 = 4.739 + B\left(\frac{0.0756}{\sqrt{50}}\right)$

CAS

$B = 1.9642$

normCDF $(-1.9642, 1.9642, 1, 0)$

$= 0.950$

$\therefore b = 95\%$ CI

✓ correct formula for B

✓ correct b (as a CI)

$5.0606 = 5.034 + 1.960\left(\frac{S_x}{\sqrt{50}}\right)$

CAS

$\therefore c = 0.09596g$

✓ correct formula for c

✓ correct c

This lawyer wanted to discredit the whole lawsuit by stating "These 100 bottles could not be from NATRAL as these two samples seem to be taken from different sources".

(d) Discuss this statement with the use of the given confidence intervals from ***Table 1***. (1 mark)

As the two 95% CI do not overlap, it can be said that they may have been taken from sources with differing means, however, we cannot be certain.

✓ correct comment (including not certain)

It was later found that Lawyer 2 chose sample A by selecting the lowest 50 scores from the original sample and chose sample B by selecting the highest 50 scores from the original sample.

(e) Explain the problem in this methodology. (1 mark)

As Lawyer 2 chose which scores went in to each sample, the samples were NOT random, hence are statistically irrelevant to make an inference.

✓ correct comment about choosing means NOT random

Question 5 **(8 marks)**

An object is traveling in a straight line. It was initially observed passing through the origin with a velocity of 4km/s. This object's velocity is always greater than 0 km/s.

The object's acceleration is defined as $a(x) = x + 4$

$t = 0$
$x = 0$
$v = 4$

(a) Show that the velocity can be defined as $v(x) = x + 4$ (3 marks)

$a = x + 4$

$v \cdot \frac{dv}{dx} = x + 4$

$\int v \, dv = \int x + 4 \, dx$

$\frac{v^2}{2} = \frac{x^2}{2} + 4x + c$

$v^2 = x^2 + 8x + c_2$

$(4)^2 = 0^2 + 8(0) + c_2$

$16 = c_2$

$v^2 = x^2 + 8x + 16$

$v = \sqrt{(x+4)^2}$

$v = x + 4$ as $v > 0$.

✓ $a(t) = v \cdot \frac{dv}{dx}$ & $\int$

✓ correct integration

✓ $v(x)$

(b) Hence, or otherwise, determine an expression for the object's displacement in terms of time, $x(t)$. (3 marks)

$v = x + 4$

$\frac{dx}{dt} = x + 4$

$\int \frac{1}{x+4} \, dx = \int 1 \, dt$

$\ln|x+4| = t + c$

$x + 4 = e^{t+c}$

$x + 4 = e^t \cdot e^c$

$x + 4 = Ae^t$

$0 + 4 = Ae^0$

$A = 4$

$\therefore x = 4e^t - 4$

✓ $v = \frac{dx}{dt}$ & $\int$

✓ $x(t)$ with c

✓ $x(t)$

For this experiment, an object is seen to be stationary when it is travelling slower than 0.000 000 01km/s $[1 \times 10^{-8}]$

(c) Determine when and from where the object likely started moving. (2 marks)
(*Hint: Do not round your working before the final answer*)

"starts" moving at

$0.000\,000\,01 = x + 4$

$x = -3.99999999$

solve $(-3.99999999 = -4 + 4e^t, t)$

$t = -19.81$ seconds

(19.81 seconds before initial observation)

✓ starting x

✓ time started moving

END OF QUESTIONS